

7.4 The Singular Value Decomposition (SVD)

So far we know two types of matrix factorization:

1) $A_{n \times n}$ has n linearly independent eigenvectors

$$A = P D P^{-1}$$

Where D diagonal and P is invertible
(eigenvalues) (eigenvectors)

2) $A_{n \times n}$ Symmetric

$$A = P D P^T$$

Where P orthogonal matrix D diagonal
(orthonormal eigenvectors) (eigenvalues).

What happens in general for an arbitrary matrix

$A_{m \times n}$ Square or not?

We will show in this section that for a general $A_{m \times n}$ matrix, there exists the following decomposition:

$$A = U \Sigma V^T$$

$$A = U \Sigma V^T$$

Where U and V are orthogonal and they may or may not be different, and Σ is "like-diagonal" for rectangular matrices.

Notice that if $A_{m \times n}$ then $U_{m \times m} \Sigma_{m \times n} V_{n \times n}$

Since

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V_{n \times n}$$

For instance, if $\Sigma_{2 \times 3}$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \rightarrow \text{"main diagonal"}$$

If $\Sigma_{3 \times 2}$ then

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix}$$

main diagonal

If $\Sigma_{5 \times 3}$ then

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

main diagonal.

Def. - For $A_{m \times n}$ matrix, we define its main diagonal to be the line of entries that starts at the upper left corner and extends diagonally as far as it can go. The entries on this main diagonal are called diagonal entries.

Important Question #1

With what do we need to construct the matrices U and V in the decomposition of any matrix $A_{m \times n}$?

$$\rightarrow A = U \Sigma V^T$$

Where $\Sigma_{m \times n}$ is like-diagonal (as shown above) for ^{non-square} rectangular matrices.

In the ^{previous} cases of square matrices, the entries of the diagonal matrix D were the eigenvalues of the matrix A , and the columns of the other factors P, P^{-1}, P^T were corresponding eigenvectors.

So, we need something similar to eigenvalues and eigenvectors for a non-square matrix $A_{m \times n}$.

Important question #2:

What would be for a non-square ^{matrix} the closest concepts to an eigenvalue and eigenvector?

The answer to this question is given analyzing the matrix

$$\boxed{A^T A}$$

Lemma. - For $A_{m \times n}$

- i) $A^T A$ is square matrix $n \times n$.
- ii) $A^T A$ is Symmetric.
- iii) $A^T A$ is orthogonally diagonalizable.
- iv) The eigenvalues λ_i of $A^T A$ are nonnegative $\lambda_i \geq 0$ $i=1, \dots, n$.

Proof. -

i) If $A_{m \times n}$ then $A_{n \times m}^T$ is an $n \times m$ matrix

$\Rightarrow A_{n \times m}^T A_{m \times n} = (A^T A)_{n \times n}$ is "square matrix."

ii) $(A^T A)^T = A^T (A^T)^T = A^T A \Rightarrow A^T A$ is Symmetric.

iii) Since $A^T A$ is Symmetric then
 $A^T A$ is orthogonally diagonalizable

In fact,

$$A^T A = P D P^T$$

Where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ and $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}$

and $\lambda_i \in \mathbb{R}$ are the eigenvalues of $A^T A$.

They may be repeated but they are real.

Also, $\vec{v}_1, \dots, \vec{v}_n$ are corresponding eigenvectors of $A^T A$. They are orthogonal and unit vectors.

It means $\vec{v}_i \cdot \vec{v}_j = 0$ if $i \neq j$
 and $\vec{v}_i \cdot \vec{v}_i = \|\vec{v}_i\|^2 = 1$.

iv) For all $i=1, \dots, n$

$$\lambda_i = \|A \vec{v}_i\|^2 \geq 0$$

In fact,

$$\begin{aligned} \|A \vec{v}_i\|^2 &= A \vec{v}_i \cdot A \vec{v}_i = (A \vec{v}_i)^T A \vec{v}_i = \vec{v}_i^T A^T A \vec{v}_i \\ &= \vec{v}_i^T (A^T A \vec{v}_i) = \vec{v}_i^T \lambda_i \vec{v}_i \\ &= \lambda_i \vec{v}_i \cdot \vec{v}_i = \lambda_i \|\vec{v}_i\|^2 = \lambda_i \end{aligned}$$

Also, $\|A \vec{v}_i\| = \sqrt{\lambda_i}, \quad i=1, \dots, n.$

Since for $A_{m \times n}$ arbitrary

$$\|A\vec{v}_i\| = \sqrt{\lambda_i} \|\vec{v}_i\|$$

It seems appropriate to define like-eigenvectors for A , the vectors $\vec{v}_i$, corresponding to a like-eigenvalue $\sigma_i = \sqrt{\lambda_i}$, $i=1, 2, \dots, n$.

Definition:

Given a matrix $A_{m \times n}$, the real values

$\sigma_i = \sqrt{\lambda_i}$, $i=1, \dots, n$, where λ_i 's are the eigenvalues of $A^T A$, are called the Singular values of A .

Also, the corresponding unit eigenvectors of λ_i are called Singular vectors of A .

Example

For the matrix $A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}_{3 \times 2}$

Find its Singular values and corresponding Singular Vectors.

Ans: By definition the singular values of A are the square root of the eigenvalues of $A^T A$, and singular vectors are any unit eigenvector $\vec{v}_i$ of $A^T A$ corresponding to an eigenvalue λ_i .

Now,

$$A^T A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Obviously symmetric as expected why?

Eigenvalues of $A^T A$:

$$\begin{aligned} |A - \lambda I| = 0 &\Leftrightarrow \begin{vmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{vmatrix} = (\lambda-9)^2 - 81 \\ &= \lambda^2 - 18\lambda = \lambda(\lambda-18) \Rightarrow \boxed{\lambda_1=18, \lambda_2=0} \end{aligned}$$

$\Rightarrow$ Singular values of A : $\sigma_1 = 3\sqrt{2}, \sigma_2 = 0$

Eigenvectors of $A^T A$:

$$\underline{\underline{\lambda_1 = 18}} \quad (A^T A - 18I) \vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} 9-18 & -9 \\ -9 & 9-18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-9x_1 - 9x_2 = 0 \Rightarrow x_1 = -x_2.$$

or $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ is a unit eigenvector of $A^T A$ and a Singular Vector of A corresponding to $\sigma_1 = \sqrt{18} = 3\sqrt{2}$.

Similarly, for

$$\underline{\underline{\lambda_2 = 0}} \quad (A^T A - 0I) \vec{x} = \vec{0} \Leftrightarrow \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or } 9x_1 = 9x_2 \Rightarrow x_1 = x_2 \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{or } \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ Singular vector of } A \text{ for } \sigma_2 = 0.$$

The vectors $A\vec{v}_i$ ($i=1, \dots, r$) for i such that $\lambda_i \neq 0$ play an important role in the SVD of A .

They are used to define the matrix U of the SVD of A

$$A = U \Sigma V^T$$

They also have very nice properties as next thm shows.

Theorem:

i) $\{\vec{v}_1, \dots, \vec{v}_n\}$ orthonormal eigenvectors of $A^T A$ (then a basis for $\mathbb{R}^n$)

ii) If corresponding eigenvalues of $A^T A$ satisfy

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \neq 0 \geq \lambda_{r+1} = 0 \geq \dots \geq \lambda_n = 0$$

Then

$\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is an orthogonal basis for $\text{Col } A$

which implies $\text{rank } A = r$.

Proof. - Recall that $\|A\vec{v}_i\| = \sqrt{\lambda_i}$,
 then if $\lambda_i \neq 0 \Rightarrow A\vec{v}_i \neq 0$, for $i=1, \dots, r$.

Also

$$\begin{aligned} A\vec{v}_i \cdot A\vec{v}_j &= \vec{v}_j^T A^T A \vec{v}_i = \vec{v}_j^T (A^T A \vec{v}_i) = \\ &= \vec{v}_j^T \lambda_i \vec{v}_i = \lambda_i \vec{v}_j^T \vec{v}_i = \lambda_i \vec{v}_i \cdot \vec{v}_j \\ &= \begin{cases} \lambda_i \|\vec{v}_i\|^2 = \lambda_i \cdot 1 = \lambda_i \\ \lambda_i \cdot 0 = 0 \end{cases} \end{aligned}$$

Therefore, $S = \{A\tilde{v}_1, \dots, A\tilde{v}_r\}$ is an orthogonal set of col A

Since none of this vectors is zero, this set is also linearly independent.

It can be shown that S also spans col A.

In fact,

For any $\tilde{y} \in \text{col } A$, there is an $x \in \mathbb{R}^n$ such that $\tilde{y} = A\tilde{x}$

and $\tilde{x} = c_1\tilde{v}_1 + \dots + c_n\tilde{v}_n$, since $\{\tilde{v}_1, \dots, \tilde{v}_n\}$ basis of $\mathbb{R}^n$.

$$\Rightarrow \tilde{y} = A\tilde{x} = c_1 A\tilde{v}_1 + \dots + c_r A\tilde{v}_r + \underbrace{\dots + c_n A\tilde{v}_n}_{=0}$$

$$\Rightarrow \tilde{y} \in \text{spans } \{A\tilde{v}_1, \dots, A\tilde{v}_r\}$$

$$\Rightarrow S = \{A\tilde{v}_1, \dots, A\tilde{v}_r\} \text{ is an orthogonal basis for col } A$$

$$\Rightarrow \text{rank } A = r.$$

Back to our objective: Decompose

matrix $A_{m \times n}$ as

$$A = U \Sigma V^T \quad (11.1)$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 & 0 \\ 0 & \sigma_2 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & \sigma_r & 0 \\ & & & 0 & 0 \end{bmatrix}_{m \times n} \quad \text{or} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 & & & \\ 0 & & \ddots & & \\ 0 & & & \sigma_r & 0 \\ 0 & & & 0 & 0 \end{bmatrix}$$

$$\text{or} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & \\ \vdots & & \ddots & \\ 0 & & & \sigma_r \\ 0 & & & 0 \end{bmatrix}_{m \times n} \quad n \geq m$$

It means Σ is the closest to a diagonal matrix for rectangular matrices.

and

$U_{m \times m}$ orthogonal matrix

$V_{n \times n}$

"

"

Determining V orthogonal matrix and Σ ("like diag" matrix)

Notice that $A_{m \times n} = U \Sigma V$

$$\Leftrightarrow A^T A = (U \Sigma V)^T (U \Sigma V) \\ = V^T \underbrace{\Sigma^T U^T U}_{=I} \Sigma V = \boxed{V^T \Sigma^T \Sigma V}$$

Also, Notice that $(\Sigma^T \Sigma)_{n \times n}$ is diagonal

and

$$\Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \sigma_r^2 \\ 0 & \dots & \dots & 0 \end{bmatrix}_{n \times n}$$

Therefore, we are looking for an orthogonal diagonalization of $A^T A$. GREAT!

We know that $A^T A$ is symmetric, as a consequence it is orthogonally diagonalizable

$$\boxed{A^T A = V D V^T}$$

where

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}_{n \times n} \quad (12.1)$$

and each $\vec{v}_i$ is an eigenvector corresponding to the eigenvalue λ_i , for all $i=1, \dots, n$.

and

$$D = \begin{bmatrix} \lambda_1 \neq 0 & 0 & \dots & 0 \\ 0 & \lambda_2 \neq 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_r \neq 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_{r+1} = 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_n = 0 \end{bmatrix}_{n \times n}$$

Comparing

$$A^T A = V^T D V \quad \text{with} \quad A^T A = V^T \Sigma^T \Sigma V$$

We arrive to the conclusion that an appropriate orthogonal matrix V for the decomposition $A = U \Sigma V$ is

$$V = [\vec{v}_1 \dots \vec{v}_r] \rightarrow \text{Normalized eigenvectors of } A^T A.$$

This forces

$$\Sigma^T \Sigma = D \Rightarrow \lambda_i = \sigma_i^2, \quad i=1, \dots, n.$$

or $\sigma_i = \sqrt{\lambda_i}, \quad i=1, \dots, n.$

As a consequence, an appropriate definition of Σ is

$$\Sigma = \begin{bmatrix} \sigma_1 = \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sigma_2 = \sqrt{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_r = \sqrt{\lambda_r} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}_{m \times n} \quad (13.1)$$

Determining V orthogonal matrix in the decomposition
 $A = V \Sigma V^T$.

Multiplying by V both ^{right} sides of the decomposition

$$AV = V \Sigma.$$

or

$$[A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_r, 0, \dots, 0] = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r, \dots, \vec{u}_m] \begin{bmatrix} \sqrt{\lambda_1} & & & & 0 \\ & \sqrt{\lambda_2} & & & \\ & & \ddots & & \\ & & & \sqrt{\lambda_r} & \\ 0 & & & & 0 & \ddots & 0 \end{bmatrix}$$

$$\Rightarrow A\vec{v}_1 = \sqrt{\lambda_1} \vec{u}_1, \dots, A\vec{v}_r = \sqrt{\lambda_r} \vec{u}_r$$

So an appropriate definition for the entries of v is

$$\boxed{u_i = \frac{1}{\sqrt{\lambda_i}} A\vec{v}_i = \frac{1}{\sigma_i} A\vec{v}_i, \quad i=1, \dots, r} \quad (14.1)$$

We have already seen that $\|A\vec{v}_i\| = \sqrt{\lambda_i} = \sigma_i$
 $i=1, \dots, r.$

$$\text{Therefore, } \|\vec{u}_i\| = \frac{\|A\vec{v}_i\|}{\sigma_i} = 1.$$

and we have also proved that

$\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is an orthogonal set of $\text{col } A$.

For the remainder U entries $\vec{u}_{r+1}, \dots, \vec{u}_m$

It is appropriate to define them as orthonormal vectors which are also orthogonal to $\vec{u}_1, \dots, \vec{u}_r$.

As a consequence, an appropriate definition of U is

$$U = \begin{bmatrix} \frac{A\vec{v}_1}{\sigma_1} & \frac{A\vec{v}_2}{\sigma_2} & \dots & \frac{A\vec{v}_r}{\sigma_r} & \vec{u}_{r+1} & \dots & \vec{u}_m \\ \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ & & & & & & \end{bmatrix}_{m \times m} \quad (15.1)$$

Bringing together (12.1), (13.1) and (15.1), we arrive to the desired decomposition:

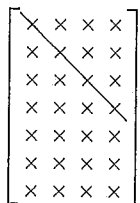
$$A = U \Sigma V^T$$

or

$$A = [\vec{u}_1 \vec{u}_2 \dots \vec{u}_r \vec{u}_{r+1} \dots \vec{u}_m] \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ 0 & & & 0 & \ddots & 0 \end{bmatrix} [\vec{v}_1 \vec{v}_2 \dots \vec{v}_n]^T \quad (15.2)$$

This special factorization of A is called

THE SINGULAR VALUE DECOMPOSITION.

Singular Value
Decomposition

Main diagonal

▲ Figure 9.5.1

Harry Bateman
(1882–1946)

Historical Note The term *singular value* is apparently due to the British-born mathematician Harry Bateman, who used it in a research paper published in 1908. Bateman emigrated to the United States in 1910, teaching at Bryn Mawr College, Johns Hopkins University, and finally at the California Institute of Technology. Interestingly, he was awarded his Ph.D. in 1913 by Johns Hopkins at which point in time he was already an eminent mathematician with 60 publications to his name.

[Image: Courtesy of the Archives, California Institute of Technology]

The vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are called the **left singular vectors** of A , and the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called the **right singular vectors** of A .

Before turning to the main result in this section, we will find it useful to extend the notion of a “main diagonal” to matrices that are not square. We define the **main diagonal** of an $m \times n$ matrix to be the line of entries shown in Figure 9.5.1—it starts at the upper left corner and extends diagonally as far as it can go. We will refer to the entries on the main diagonal as the **diagonal entries**.

We are now ready to consider the main result in this section, which is concerned with a specific way of factoring a general $m \times n$ matrix A . This factorization, called **singular value decomposition** (abbreviated SVD) will be given in two forms, a brief form that captures the main idea, and an expanded form that spells out the details. The proof is given at the end of this section.

THEOREM 9.5.3 Singular Value Decomposition

If A is an $m \times n$ matrix, then A can be expressed in the form

$$A = U\Sigma V^T$$

where U and V are orthogonal matrices and Σ is an $m \times n$ matrix whose diagonal entries are the singular values of A and whose other entries are zero.

THEOREM 9.5.4 Singular Value Decomposition (Expanded Form)

If A is an $m \times n$ matrix of rank k , then A can be factored as

$$A = U\Sigma V^T = [\mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_k \mid \mathbf{u}_{k+1} \ \cdots \ \mathbf{u}_m] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & & \\ 0 & \sigma_2 & \cdots & 0 & & \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & \cdots & \sigma_k & & \\ \hline & & & 0_{(m-k) \times k} & 0_{(m-k) \times (n-k)} & \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_k^T \\ \hline \mathbf{v}_{k+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}$$

in which U , Σ , and V have sizes $m \times m$, $m \times n$, and $n \times n$, respectively, and in which

- $V = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$ orthogonally diagonalizes $A^T A$.
- The nonzero diagonal entries of Σ are $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $\dots$, $\sigma_k = \sqrt{\lambda_k}$, where $\lambda_1, \lambda_2, \dots, \lambda_k$ are the nonzero eigenvalues of $A^T A$ corresponding to the column vectors of V .
- The column vectors of V are ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$.
- $\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \frac{1}{\sigma_i} A\mathbf{v}_i \quad (i = 1, 2, \dots, k)$
- $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is an orthonormal basis for $\text{col}(A)$.
- $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_m\}$ is an extension of $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ to an orthonormal basis for R^m .

Example Find a Singular Value Decomposition of

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$$

Ans:

We already found
Eigenvalues and corresponding ^{unit} eigenvectors for

$A^T A$. They are $\lambda_1 = 18 \rightarrow \vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\lambda_2 = 0 \rightarrow \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Notice that $\vec{v}_1 \cdot \vec{v}_2 = 0$ as expected.

$$A\vec{v}_1 = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4/\sqrt{2} \\ 2/\sqrt{2} \\ -4/\sqrt{2} \end{bmatrix}$$

$$\|A\vec{v}_1\| = \sqrt{\lambda_1} = \sqrt{18} = 3\sqrt{2}$$

$$\Rightarrow \vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

Since, an SVD of A is given by

$$A = U \Sigma V^T,$$

So far we have

$$A_{3 \times 2} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ 2/3 & ? & ? \\ 1/3 & ? & ? \\ -2/3 & ? & ? \end{bmatrix}_{3 \times 3} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3 \times 2} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{2 \times 2}$$

$U \quad \Sigma \quad V^T$

We still need $\vec{u}_2$ and $\vec{u}_3$ unit vectors and

$$\vec{u}_2 \perp \vec{u}_1, \quad \vec{u}_3 \perp \vec{u}_1, \quad \text{and} \quad \vec{u}_3 \perp \vec{u}_2$$

$$\vec{u}_2 \cdot \vec{u}_1 = 0 \quad \vec{u}_3 \cdot \vec{u}_1 = 0 \quad \vec{u}_3 \cdot \vec{u}_2 = 0$$

Any vector $\vec{w}$ orthogonal to $\vec{u}_1$ should satisfy

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\vec{w} \cdot \vec{u}_1 = 0 \Leftrightarrow \frac{2}{3}w_1 + \frac{1}{3}w_2 - \frac{2}{3}w_3 = 0 \Leftrightarrow$$

$$\vec{w}_1 = w_3 - \frac{1}{2}w_2 \quad \text{or} \quad \vec{w} = \begin{bmatrix} w_3 - \frac{1}{2}w_2 \\ w_2 \\ w_3 \end{bmatrix} =$$

$$= w_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis in $\mathbb{R}^3$ for

vectors $\vec{w}$ orthogonal to $\vec{u}_1$.

Since, they are not orthogonal ^{between them,} we apply G-S.

to obtain an orthogonal basis.

We start defining, $\vec{u}_2' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \vec{u}_3' = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \cdot \vec{u}_2'}{\vec{u}_2' \cdot \vec{u}_2'} \vec{u}_2' =$

$$= \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{(-1/2)}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

$$\Rightarrow u_2 = \frac{\vec{u}_2'}{\|\vec{u}_2'\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad u_3 = \frac{\vec{u}_3'}{\|\vec{u}_3'\|} = \frac{4}{3\sqrt{2}} \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/6 \\ 2\sqrt{2}/3 \\ \sqrt{2}/6 \end{bmatrix}$$

Since, $\|\vec{u}_3'\| = \sqrt{\frac{1}{16} + 1 + \frac{1}{16}} = \sqrt{\frac{18}{16}} = \frac{3\sqrt{2}}{4}$

and finally.

$$A = \begin{matrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ \begin{bmatrix} 2/3 & 1/\sqrt{2} & -\sqrt{2}/6 \\ 1/3 & 0 & 2\sqrt{2}/3 \\ -2/3 & 1/\sqrt{2} & \sqrt{2}/6 \end{bmatrix} \end{matrix} \begin{matrix} \vec{\sigma}_1 = 3\sqrt{2} \\ \vec{\sigma}_2 \\ 0 \end{matrix} \begin{matrix} \vec{v}_1 & \vec{v}_2 \\ \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \end{matrix}$$

$$U \Sigma V^T$$

Verify that A is actually equal to

$$U \Sigma V^T.$$